Sampling for Graph Colorings: An Alternative Variable Length Coupling Approach

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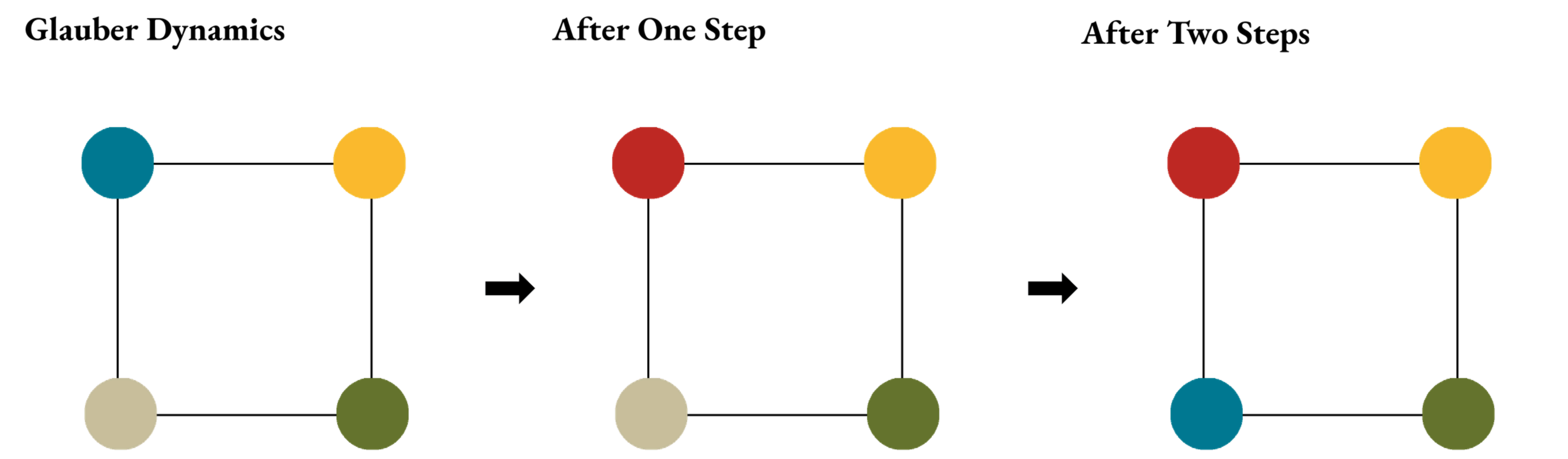
Sampling for Colorings

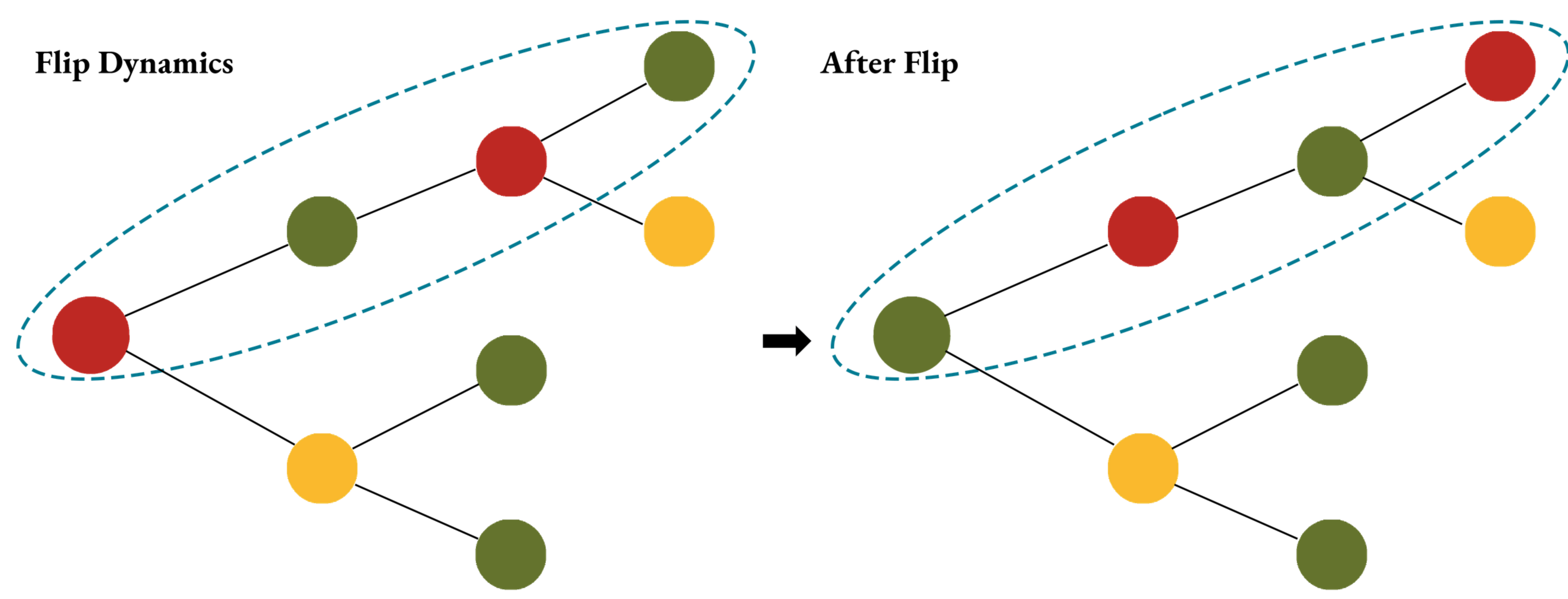
k **-coloring:** Assignment of one of k labels to each vertex so no neighbors have the same label. Straightforward when $k \geq \Delta + 1$ due to extra color.

Goal: Generate a uniformly random coloring in polynomial time on general graphs with maximum degree Δ. Note, the number of colorings is exponential in the number of vertices.

Problem: For what values of k is it possible to do in polynomial time?

Dynamics





Glauber Dynamics:

1. Randomly select vertex and color.
2. Attempt to set vertex to that color.
3. If valid coloring, continue, otherwise, revert.

Flip Dynamics:

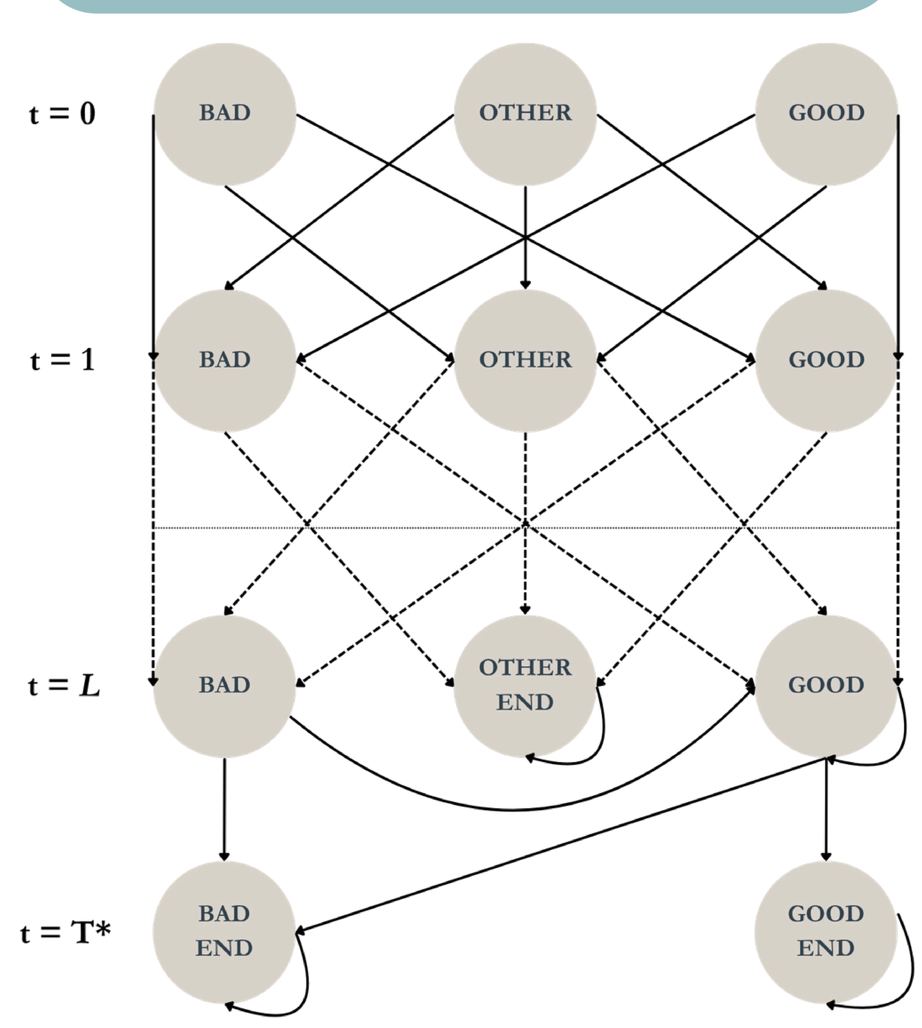
1. Randomly select vertex and color c.
2. Attempt to "flip" maximal two-colored component rooted at that vertex with color c. Flip with probability relative to its length.

Relevant Results

Paper	Result (k)	Flip / Glauber	Analysis
Jerrum '95	2Δ	Glauber	Simple
Vigoda '99	$(11/6)\Delta$	Flip	Metric
Chen et al. '19	1.83332Δ	Flip	Variable Length*
Carlson, Vigoda '25	1.809Δ	Flip	Metric
Our Work	**1.832Δ**	**Flip**	**Variable Length**

*: alternative proof with metric analysis.

States and Transitions



A visualization of the state machine developed for the layered approach. The chain could start in either Good/Bad/Other and transition between each other. L is the number of layers (transitions) and T^* denotes the time that the Hamming distance changes. Dotted lines imply many (L - 1) layers that are not shown in the diagram. See description to the right for more details.

Our Variable Length Approach

Coupling: Technique to analyze mixing time (time to stationary, π) using two chains, Xt and Yt, by bounding the time until they meet.

$$||X_t - \pi||_{TV} \leq \Pr(X_t \neq Y_t) \leq \frac{1}{4}$$

Variable Length Coupling: A path coupling approach that utilizes multi-step analysis (rather than one step). In this case, consider until the Hamming distance changes.

Layered Approach: Previous approaches didn't allow for back-and-forth transitions between states. Layered approach unrolls transitions with no restrictions for L steps.

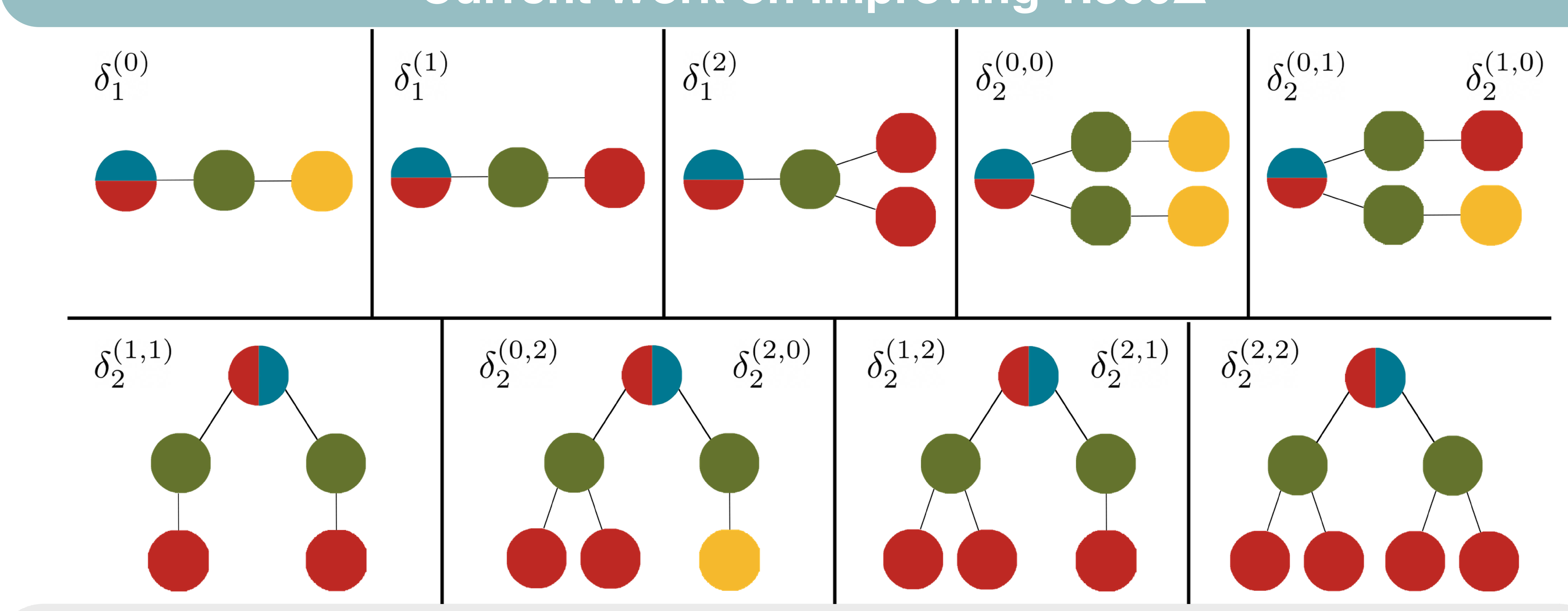
States:

- **Good:** Likely to decrease distance and converge.
- **Bad:** Likely to increase distance and diverge.
- **Other:** Remaining states with less of an impact.
- **End:** Analog after Hamming distance has changed.

Conclusion

Main Result: Our analysis shows polynomial time mixing when $k > 1.832\Delta$.
Takeaways: Layered transitions can improve variable length methods.
Limitations: Observed bottleneck with variable length methods that prevents the capture of "tradeoffs," something captured well by a metric analysis.

Current Work on Improving 1.809Δ



Future direction indicates improved analysis with metric could near 1.8Δ through:

- Tightening the probability of the disagreement vertex changing colors.
- Considering cost of neighbor "types" flipping and related extremal cases.
- Introducing a more complex version of the metric to encourage transitions between adjacent neighborhood types, further categorizing Good and Bad configurations.

Acknowledgment

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